Chaotic one-dimensional harmonic oscillator

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We present one of the simplest systems that exhibit resonance overlap and chaos: a nonrelativistic one-dimensional simple harmonic oscillator driven by a space-time varying force. The theoretical study of nonlinear resonances can be carried out analytically using second-order canonical perturbation theory. [S1063-651X(97)05411-1]

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I. INTRODUCTION

The simple harmonic oscillator represents a standard linear system that always behaves regularly even when a time-varying external force is present. It has been shown recently, however, that the simple harmonic oscillator driven by a time-periodic force can exhibit chaos if the motion becomes relativistic [1]. In this paper we show that the simple harmonic oscillator, even in the nonrelativisic regime, can behave chaotically if it is driven by an external force that varies in both time and space. A special type of space-time–dependent force that varies periodically both in time and in space applied to a simple harmonic oscillator is encountered frequently in plasma physics in relation to the cyclotron motion of a charged particle interacting with an electromagnetic wave [2–5]. Here we consider a more general class of a space-time–varying force and show, using canonical perturbation theory [6], that the force generates a series of nonlinear resonances in the phase space of the oscillator. As is well known, when the neighboring resonances overlap, chaos occurs.

Since the simple harmonic oscillator is a linear system whose period of oscillation is independent of energy, one must go beyond first-order perturbation theory in order to show that nonlinear resonances are generated. Yet, due to the simplicity of the system, calculations can be carried out analytically without difficulty, at least to second order of perturbation. The significance of the system described in the present work is that it is one of the simplest systems that exhibit resonance overlap and chaos and that it represents a system on which high-order canonical perturbation theory can be performed analytically.

II. SYSTEM

Let us consider a nonrelativistic one-dimensional simple harmonic oscillator of mass μ and frequency w_0 driven by an external force $F(q,t)$ that varies with the spatial coordinate q as well as with time t. The Hamiltonian for the oscillator can be written as

$$H(q,p,t)=\frac{p^2}{2\mu}+\frac{1}{2}\mu w_0^2 q^2-\int^q F(q',t)dq'. \tag{1}$$

We limit ourselves to the case where the force $F(q,t)$ is periodic in time t with frequency $w=2\pi/T$. We also assume that $F(q,t)$ is an even function of t and that $\int_0^T F(q,t)dt=0$. We can then express $F(q,t)$ in Fourier series as

$$F(q,t)=F_0\sum_{n=1}^{\infty} a_n(q)\cos nwt, \tag{2}$$

where F_0 is a constant that measures the strength of the applied force.

In order to apply canonical perturbation theory to the system being considered, it is convenient to go to the action-angle space. For the simple harmonic oscillator the action and angle variables I,θ are related to q,p by [7]

$$q=(2I/\mu w_0)^{1/2}\cos\theta,\quad p=-(2\mu w_0 I)^{1/2}\sin\theta. \tag{3}$$

Substituting Eqs. (2) and (3) into Eq. (1), we can express the Hamiltonian as

$$\begin{aligned}H(I,\theta,t)&=w_0 I-F_0\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\alpha_{nm}(I)\cos m\theta\cos nwt\\&=w_0 I-\frac{F_0}{2}\sum_n\sum_m \alpha_{nm}(I)[\cos(m\theta+nwt)\\&\quad+\cos(m\theta-nwt)],\end{aligned} \tag{4}$$

where the coefficients $\alpha_{nm}(I)$ can be determined from a_n's.

Equation (4) indicates that nonlinear resonances occur if the condition

$$m\frac{d\theta}{dt}=nw \tag{5}$$

is satisfied. In the standard perturbation treatment of the dynamics of a driven oscillator, the quantity $d\theta/dt$ is usually taken as the frequency in the lowest order of perturbation, i.e., as the frequency of oscillation in the absence of an external force. Equation (5) then becomes, for our case of the simple harmonic oscillator,

$$mw_0=nw. \tag{6}$$

Equation (6) is totally independent of the oscillator energy and thus the formation of nonlinear resonances in different regions of phase space is not indicated. One should keep in mind, however, that the frequency of oscillation in the presence of an external force is in general different from that in the absence of an external force. We show below that, even for our case of the simple harmonic oscillator, application of a spatially varying external force indeed shifts the frequency of oscillation from its natural value w_0. In order to show that, we need to carry canonical perturbation theory at least to second order.

III. SECOND-ORDER PERTURBATION THEORY

In the canonical perturbation theory [6], one seeks a canonical transformation from the usual action-angle variables (I,θ) to a new set of action-angle variables (J,ϕ), which allows an identification of an invariant to a desired order of perturbation. The generating function for the desired transformation is written as

$$S(J,\theta,t)=J\theta+\epsilon S_1(J,\theta,t)+\epsilon^2 S_2(J,\theta,t)+\cdots, \quad (7)$$

where the Hamiltonian of Eq. (4) is now written as

$$H(I,\theta,t)=w_0 I-\epsilon\frac{F_0}{2}\sum_n\sum_m \alpha_{nm}(I)[\cos(m\theta+nwt)+\cos(m\theta-nwt)]. \quad (8)$$

ϵ is a parameter introduced to identify the driving force terms as the perturbation and will be set $\epsilon=1$ at the end of the calculation. The relation between the old and new sets of the action-angle variables is given by

$$I=\frac{\partial S}{\partial\theta}=J+\epsilon\frac{\partial S_1(J,\theta,t)}{\partial\theta}+\epsilon^2\frac{\partial S_2(J,\theta,t)}{\partial\theta}+\cdots, \quad (9)$$

$$\phi=\frac{\partial S}{\partial J}=\theta+\epsilon\frac{\partial S_1(J,\theta,t)}{\partial J}+\epsilon^2\frac{\partial S_2(J,\theta,t)}{\partial J}+\cdots \quad (10)$$

and the transformed Hamiltonian is

$$K(J,\phi,t)=H(I,\theta,t)+\frac{\partial S(J,\theta,t)}{\partial t}. \quad (11)$$

Substituting Eqs. (7) and (8) into Eq. (11) and utilizing Eqs. (9) and (10) to collect terms of the same order in ϵ, we obtain

$$K(J,\phi,t)=w_0J+\epsilon\left\{w_0\frac{\partial S_1}{\partial\theta}+\frac{\partial S_1}{\partial t}-\frac{F_0}{2}\sum_n\sum_m \alpha_{nm}(J)\times[\cos(m\theta+nwt)+\cos(m\theta-nwt)]\right\}+\epsilon^2\left\{w_0\frac{\partial S_2}{\partial\theta}+\frac{\partial S_2}{\partial t}-\frac{F_0}{2}\sum_n\sum_m\frac{d\alpha_{nm}(J)}{dJ}\frac{\partial S_1}{\partial\theta}[\cos(m\theta+nwt)+\cos(m\theta-nwt)]\right\}+\cdots. \quad (12)$$

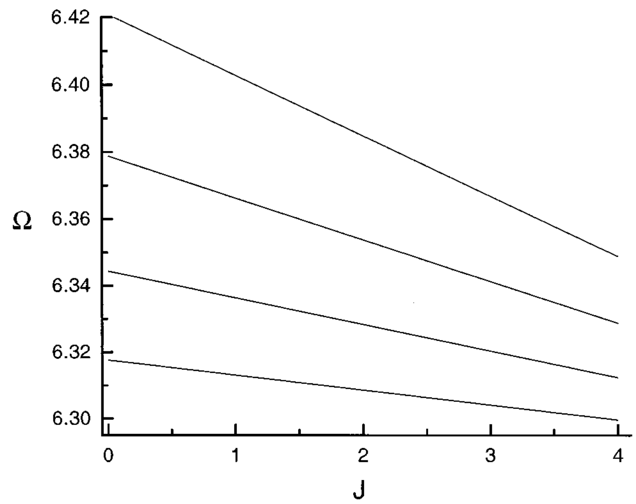

FIG. 1. Frequency of oscillation Ω as a function of J. The curves from bottom to top correspond to the case where $F_0=3$, 4, 5, and 6 (in arbitrary units), respectively. The parameters are $\mu=1$, $w_0=2\pi$, and $w=1$ (in arbitrary units).

The generating functions $S_1(J,\theta,t)$ and $S_2(J,\theta,t)$ are now to be chosen to eliminate the θ- and t-dependent parts of K to first and second order, respectively, in ϵ. We thus have, for S_1,

$$w_0\frac{\partial S_1(J,\theta,t)}{\partial\theta}+\frac{\partial S_1(J,\theta,t)}{\partial t}=\frac{F_0}{2}\sum_n\sum_m\alpha_{nm}(J)[\cos(m\theta+nwt)+\cos(m\theta-nwt)]-\frac{F_0}{2}\left\langle\sum_n\sum_m\alpha_{nm}(J)\times[\cos(m\theta+nwt)+\cos(m\theta-nwt)]\right\rangle, \quad (13)$$

where the angular brackets denote a quantity averaged over θ and t, i.e.,

$$\langle f(\theta,t)\rangle=\frac{1}{2\pi}\int_0^{2\pi}d\theta\frac{1}{T}\int_0^T dt\ f(\theta,t). \quad (14)$$

Noting that the last term on the right-hand side of Eq. (13) vanishes, we can solve Eq. (13) for S_1 and obtain

$$S_1(J,\theta,t)=\frac{F_0}{2}\sum_n\sum_m\alpha_{nm}(J)\left[\frac{\sin(m\theta+nwt)}{mw_0+nw}+\frac{\sin(m\theta-nwt)}{mw_0-nw}\right]. \quad (15)$$

Substituting Eq. (15) into Eq. (12) and requiring the term proportional to ϵ^2 be independent of θ and t, we can similarly determine $S_2(J,\theta,t)$. With S_1 and S_2 determined as above, the Hamiltonian given by Eq. (12) can now be written as

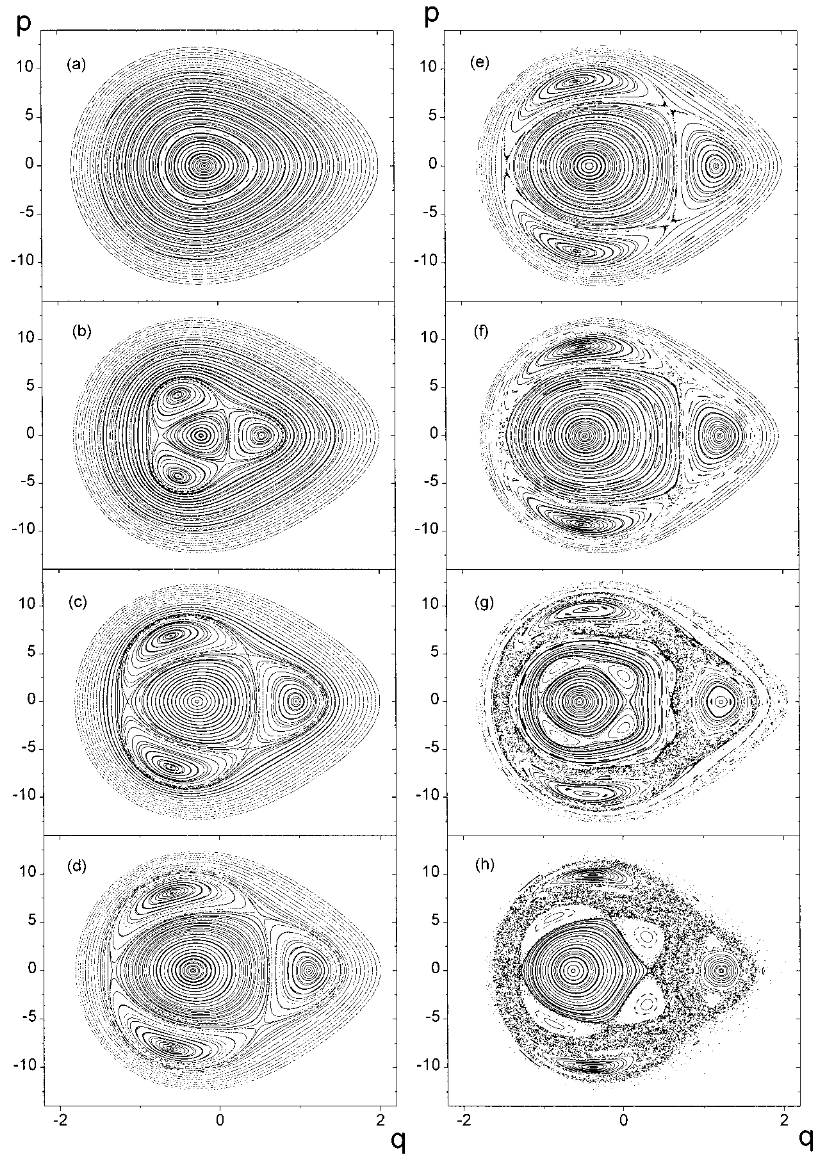

FIG. 2. Poincaré phase-space plots for the case where F_0 is (a) 3, (b) 4, (c) 5, (d) 6, (e) 7, (f) 8, (g) 9, and (h) 10 (in arbitrary units). The parameters are $\mu=1$, $w_0=2\pi$, and $w=1$ (in arbitrary units).

$$K(J,\phi,t)=w_0J+\epsilon K_1(J)+\epsilon^2K_2(J)+\epsilon^3K_3(J,\phi,t)+\cdots, \tag{16}$$

where

$$K_1(J)=-\frac{F_0}{2}\left\langle \sum_n \sum_m \alpha_{nm}(J)[\cos(m\theta+nwt) +\cos(m\theta-nwt)]\right\rangle =0 \tag{17}$$

and

$$K_2(J)=-\frac{F_0}{2}\left\langle \sum_n \sum_m \frac{d\alpha_{nm}(J)}{dJ}\frac{\partial S_1(J,\theta,t)}{\partial\theta}[\cos(m\theta+nwt) +\cos(m\theta-nwt)]\right\rangle. \tag{18}$$

Substitution of Eq. (15) into Eq. (18) yields

$$K_2(J)=-\frac{F_0^2}{8}\sum_n \sum_m \frac{d[\alpha_{nm}(J)]^2}{dJ}\frac{m^2w_0}{(mw_0+nw)(mw_0-nw)}. \tag{19}$$

We note that, to second order in ϵ, the new action variable J is a constant of motion. The frequency of oscillation in the presence of an external force, which we denote by Ω, can now be determined, to second order in ϵ, by

$$\Omega(J)=\frac{\partial K}{\partial J}=w_0+\epsilon\frac{\partial K_1}{\partial J}+\epsilon^2\frac{\partial K_2}{\partial J}. \tag{20}$$

Substituting Eqs. (17) and (19) into Eq. (20) and setting $\epsilon=1$, we obtain

$$\Omega(J)=w_0-\frac{F_0^2}{8}\sum_n\sum_m\frac{d^2[\alpha_{nm}(J)]^2}{dJ^2}\times\frac{m^2w_0}{(mw_0+nw)(mw_0-nw)}. \tag{21}$$

The resonance condition (5) now becomes

$$m\Omega(J)=nw. \tag{22}$$

Since the frequency Ω varies with J and thus with the oscillator energy, nonlinear resonances can be generated in different regions of phase space and chaotic motion can occur when neighboring resonances overlap. We illustrate this with an example in Sec. IV.

Before closing this section, we consider briefly the simple case when the driving force is independent of the spatial coordinate. Taking $F=F_0\cos wt$, we have

$$H=\frac{p^2}{2\mu}+\frac{1}{2}\mu w_0^2q^2-qF_0\cos wt$$
$$=w_0I-F_0\left(\frac{2I}{\mu w_0}\right)^{1/2}\cos\theta\cos wt. \tag{23}$$

A comparison with Eq. (4) immediately yields

$$\alpha_{nm}(I)=(2I/\mu w_0)^{1/2}\delta_{n1}\delta_{m1}. \tag{24}$$

Since $d^2[\alpha_{11}(J)]^2/dJ^2=0$, we obtain, from Eq. (21),

$$\Omega(J)=w_0. \tag{25}$$

Equation (25) indicates that, when the driving force is independent of the spatial coordinate, the frequency of the driven simple harmonic oscillator is the same as the natural frequency of the undriven oscillator and thus nonlinear resonances cannot be generated.

IV. EXAMPLE

As an illustration of a spatially varying time-periodic force that leads to generation of nonlinear resonances and eventually to chaotic motion of a simple harmonic oscillator, we take

$$F(q,t)=-F_0(1-q^2)u(t), \tag{26}$$

where $u(t)$ represents a square wave in time t given by

$$u(t)=\begin{cases}1 & \text{if } 2n\pi-\frac{\pi}{2}\leqslant t\leqslant 2n\pi+\frac{\pi}{2}\\ -1 & \text{otherwise.}\end{cases} \tag{27}$$

The function $u(t)$ can be expanded in Fourier series as

$$u(t)=\frac{4}{\pi}\left[\cos wt-\frac{1}{3}\cos 3wt+\frac{1}{5}\cos 5wt+\cdots\right]$$
$$=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\cos(2n-1)wt. \tag{28}$$

The Hamiltonian for the oscillator is given by

$$H(q,p,t)=\frac{p^2}{2\mu}+\frac{1}{2}\mu w_0^2q^2+F_0\left(q-\frac{q^3}{3}\right)u(t)$$
$$=w_0I+F_0\left\{\left[\left(\frac{2I}{\mu w_0}\right)^{1/2}-\frac{1}{4}\left(\frac{2I}{\mu w_0}\right)^{3/2}\right]\cos\theta-\frac{1}{12}\left(\frac{2I}{\mu w_0}\right)^{3/2}\cos 3\theta\right\}u(t). \tag{29}$$

Comparing Eq. (29) with Eq. (4), we obtain

$$\alpha_{(2n-1)1}(t)=-\left[\left(\frac{2I}{\mu w_0}\right)^{1/2}-\frac{1}{4}\left(\frac{2I}{\mu w_0}\right)^{3/2}\right]\frac{4}{\pi}\frac{(-1)^{n+1}}{2n-1}, \tag{30}$$

$$\alpha_{(2n-1)3}(t)=\frac{1}{12}\left(\frac{2I}{\mu w_0}\right)^{3/2}\frac{4}{\pi}\frac{(-1)^{n+1}}{2n-1}, \tag{31}$$

$$\alpha_{(2n)1}(t)=\alpha_{(2n)3}(t)=0, \tag{32}$$

$$\alpha_{n2}(t)=\alpha_{n4}(t)=\alpha_{n5}(t)=\alpha_{n6}(t)=\cdots=0. \tag{33}$$

The generating function S_1 is then determined by Eq. (15) as

$$S_1(J,\theta,t)=-\frac{2F_0}{\pi}\left[\left(\frac{2J}{\mu w_0}\right)^{1/2}-\frac{1}{4}\left(\frac{2J}{\mu w_0}\right)^{3/2}\right]\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\left\{\frac{\sin[(\theta+(2n-1)wt]}{w_0+(2n-1)w}+\frac{\sin[\theta-(2n-1)wt]}{w_0-(2n-1)w}\right\}$$
$$+\frac{F_0}{6\pi}\left(\frac{2J}{\mu w_0}\right)^{3/2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\left\{\frac{\sin[3\theta+(2n-1)wt]}{3w_0+(2n-1)w}+\frac{\sin[3\theta-(2n-1)wt]}{3w_0-(2n-1)w}\right\}. \tag{34}$$

Finally, the frequency $\Omega(J)$ can be obtained by substituting Eqs. (30)–(33) into Eq. (21). We have

$$\Omega(J)=w_0-\frac{2F_0^2}{\pi^2}\left\{\frac{\partial^2}{\partial J^2}\left[\left(\frac{2J}{\mu w_0}\right)^{1/2}-\frac{1}{4}\left(\frac{2J}{\mu w_0}\right)^{3/2}\right]^2\sum_n\frac{1}{(2n-1)^2}\frac{w_0}{[w_0+(2n-1)w][w_0-(2n-1)w]}\right.$$
$$\left.+\frac{1}{12^2}\frac{\partial^2}{\partial J^2}\left(\frac{2J}{\mu w_0}\right)^3\sum_n\frac{1}{(2n-1)^2}\frac{9w_0}{[3w_0+(2n-1)w][3w_0-(2n-1)w]}\right\}. \tag{35}$$

TABLE I. Location of the elliptic and hyperbolic fixed points of the $6\frac{1}{3}$ resonance for various values of F_0, determined theoretically using the second-order perturbation theory and numerically from the computed Poincaré plots. Only the fixed points whose p value is zero are considered.

	Theoretical		Numerical	
F_0	Elliptic	Hyperbolic	Elliptic	Hyperbolic
4	0.59	−0.73	0.54	−0.70
5	1.01	−1.14	0.96	−1.15
6	1.18	−1.30	1.11	−1.33
7	1.27	−1.40	1.19	−1.44
8	1.33	−1.45	1.22	

In Fig. 1 we show Ω as a function of J for different values of F_0. The parameter values chosen for our computation are $\mu=1$, $w_0=2\pi$, and $w=1$ (in arbitrary units). We note that for the parameter values chosen as above and for the range of J considered, high-order terms in Eq. (35) can be neglected. In fact, it is usually sufficient to consider only the first two terms ($n=1$ and 2) in the first series and the first term ($n=1$) in the second series on the right-hand side of Eq. (35). Figure 1 indicates that the resonance condition $\Omega/w=n/m=6\frac{1}{3}$ can be met if F_0 is greater than a critical value that lies between 3 and 4. The value of J at which that resonance condition is satisfied increases as F_0 is increased beyond the critical value. We mention here that with the ratio $w_0/w=2\pi$ as chosen above, the resonance corresponding to the condition $\Omega/w=6\frac{1}{3}$ is the first one to appear when an external force is applied.

In Fig. 2 we show Poincaré maps obtained by numerically integrating Hamilton's equations of motion for the driven simple harmonic oscillator being considered and by marking the phase-space positions of the oscillator at $t=nT=2\pi n/w$. The resonance islands corresponding to $\Omega/w=6\frac{1}{3}$ that are missing at $F_0=3$ are clearly seen at $F_0=4$ and at higher values of F_0. It can also be observed that, as F_0 is increased, the elliptic fixed points of the resonance move outward from the origin. One can also see clearly that at $F_0=8$ resonance islands corresponding to $\Omega/w=6\frac{3}{7}$ appear and that at $F_0=9$ those corresponding to $\Omega/w=6\frac{1}{2}$ are generated. It is evident that at $F_0=9$ the $6\frac{1}{3}$ and $6\frac{3}{7}$ resonances overlap and produce the region of chaos. This chaotic sea is seen to occupy a large portion of the phase space at $F_0=10$.

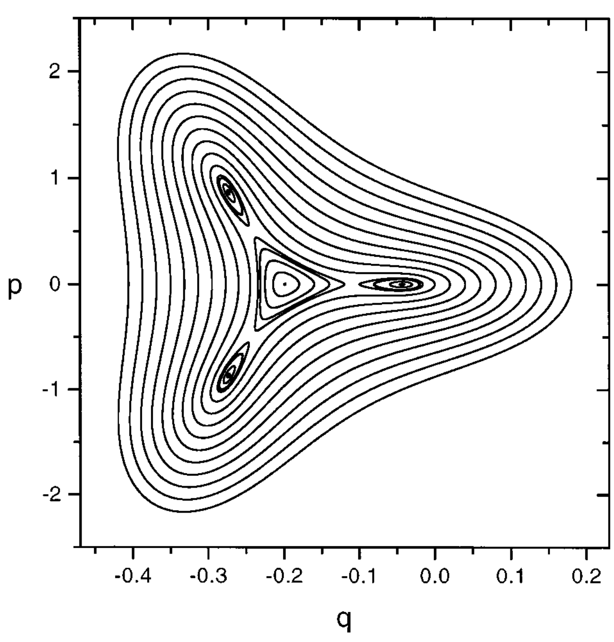

FIG. 3. Poincaré phase-space plot at $F_0=3.665$ (in arbitrary units) at which the $6\frac{1}{3}$ resonance is generated.

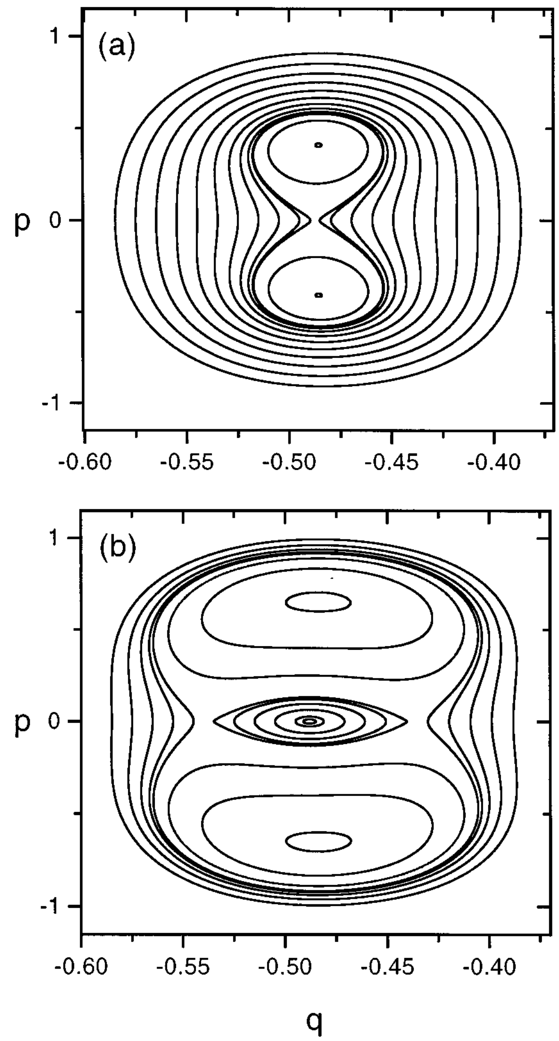

FIG. 4. Poincaré phase-space plots at (a) $F_0=8.29$ and (b) $F_0=8.3$ (in arbitrary units) at which the $6\frac{1}{2}$ resonance is generated.

The location of the fixed points of the resonances can be estimated by straightforward algebra as follows. For a given value of F_0 we first solve Eq. (35) to obtain the value of J that yields Ω corresponding to the resonance being considered, say $\Omega=6\frac{1}{3}$ if the fixed points of the $6\frac{1}{3}$ resonance are to be determined. We then substitute this value of J into Eq. (9), take $t=2\pi n/w$, and assign an appropriate value of θ, which allows determination of the value of I corresponding to the fixed point. If, for example, the position of the elliptic (hyperbolic) fixed point of the $6\frac{1}{3}$ resonance lying on the positive (negative) q axis of Fig. 2 is to be determined, the value $\theta=0$ (π) should be chosen. The locations of the elliptic and hyperbolic fixed points calculated as above are shown in Table I for the $6\frac{1}{3}$ resonance along with those determined numerically. The agreement between the two sets is seen to be reasonably good.

In Figs. 3 and 4 we present Poincaré maps at the birth of the $6\frac{1}{3}$ and $6\frac{1}{2}$ resonances, respectively. The elliptic and hyperbolic fixed points of the $6\frac{1}{3}$ resonance are generated via saddle-node bifurcations, whereas those of the $6\frac{1}{2}$ resonance are generated via period-doubling bifurcations. These bifurcation sequences as described in the figures are consistent with the theorem of Meyer [8,9].

V. CONCLUSION AND DISCUSSION

We have shown that the nonrelativistic one-dimensional simple harmonic oscillator driven by a space-time dependent force can exhibit resonance overlap and chaos. This represents one of the simplest systems known to exhibit chaos.

It should be emphasized that in order for an oscillator to show chaotic behavior arising from resonance overlap, nonlinear resonances should be generated in different phase-space regions of the oscillator, which in turn requires that the oscillation frequency varies with respect to energy. The reason why the time-driven simple harmonic oscillator can behave chaotically when it moves at relativistic velocities is because its frequency is no longer constant in the relativistic region [1]. In the nonrelativistic region, the oscillation frequency of a simple harmonic oscillator remains constant, even if a time-varying force is applied. What we have shown in this work is that the frequency of the nonrelativistic simple harmonic oscillator is shifted and becomes energy dependent in the presence of a spatially varying external force. The frequency shift of the same nature occurs when a time-varying force is applied to the relativistic oscillator of constant period [10]. A proper treatment of the frequency shift and consequently of the nonlinear resonances and chaos occurring in the present system requires canonical perturbation theory to be carried at least to second order. Calculations required, however, are all straightforward and can be performed using elementary functions only.

ACKNOWLEDGMENTS

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[1] J. H. Kim and H. W. Lee, Phys. Rev. E **51**, 1579 (1995).
[2] G. M. Zaslavsly, R. Z. Sagdeev, D. A. Usikov, and A. A. Chernikov, *Weak Chaos and Quasi-Regular Patterns* (Cambridge University Press, Cambridge, 1991), pp. 90–95.
[3] A. Fukuyama, H. Momota, R. Itatani, and T. Takizuka, Phys. Rev. Lett. **38**, 701 (1977).
[4] C. F. F. Karney, Phys. Fluids **21**, 1584 (1978).
[5] G. M. Zaslavsky, M. Y. Zakharov, R. Z. Sagdeev, D. A. Usikov, and A. A. Chernikov, Sov. Phys. JETP **64**, 294 (1986).
[6] A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, New York, 1983), Chap. 2.
[7] H. Goldstein, *Classical Mechaincs*, 2nd ed. (Addison-Wesley, Reading, MA, 1981), p. 462.
[8] K. R. Meyer, Trans. Am. Math. Soc. **149**, 95 (1970).
[9] D. A. Sadovskií and J. B. Delos, Phys. Rev. E **54**, 2033 (1996).
[10] J. H. Kim, S. W. Lee, H. Maassen, and H. W. Lee, Phys. Rev. A **53**, 2991 (1996); S. W. Lee, J. H. Kim, and H. W. Lee, Phys. Rev. E **56**, 4090 (1997).